Section 3: Introduction to Functions
Section 3 – Topic 1
Input and Output Values

A function is a relationship between input and output.

- **Domain** is the set of values of \( x \) used for the **input** of the function.

- **Range** is the set of values of \( y \) calculated from the domain for the **output** of the function.

In a function, every \( x \) corresponds to **only** one \( y \).

- \( y \) can also be written as \( f(x) \).

Consider the following function.

```
For every \( x \) there is a unique \( y \).

\[ \begin{array}{c}
1 & \rightarrow & 25 \\
2 & \rightarrow & 50 \\
3 & \rightarrow & 75 \\
\end{array} \]
```

input \( \rightarrow \) output

domain \( \rightarrow \) range
We also refer to the variables as independent and dependent. The dependent variable depends on the independent variable.

Refer to the mapping diagram on the previous page.

Which variable is independent? \( x \)

Which variable is dependent? \( y \)

Consider a square whose perimeter depends on the length of its sides.

What is the independent variable?

**The length of the sides**

What is the dependent variable?

**The perimeter**

How can you represent this situation using function notation?

Let \( l \) represent the length of one side.

\[ f(l) = 4l \]

**Tip:**
We can choose any letter to represent a function, such as \( f(x) \) or \( g(x) \), where \( x \) is the input value. By using different letters, we show that we are talking about different functions.
1. You earn $10.00 per hour babysitting. Your total earnings depend on the number of hours you spend babysitting.

a. What is the independent variable?

Number of hours spent babysitting

b. What is the dependent variable?

Total earnings

c. How would you represent this situation using function notation?

Let $h$ represent number of hours spent babysitting.

$g(h) = 10h$

2. The table below represents a relation.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>$-3$</td>
<td>8</td>
</tr>
</tbody>
</table>

a. Is the relation also a function? Justify your answer.

No, the input value $-3$ gives two different output values, 5 and 8.

b. If the relation is not a function, what number could be changed to make it a function?

We could change one of the $-3$ to anything other than 0 or 2.

a. What does her total cost depend upon?

   The number of composition books she buys

b. What are the input and output?

   Input – The number of composition books
   Output – The total cost

c. Write a function to describe the situation.

   Let \( c \) represent the number of composition books.
   \[ h(c) = 1.25c \]

d. If Mrs. Krabappel buys 24 composition books, they will cost her $30.00. Write this function using function notation.

   \[ h(24) = 1.25(24) = 30 \]
4. Consider the following incomplete mapping diagrams.

Diagram A

Diagram B

Diagram C

a. Complete Diagram A so that it is a function.

See diagram.

b. Complete Diagram B so that it is NOT a function.

See diagram.

c. Is it possible to complete the mapping diagram for Diagram C so it represents a function? If so, complete the diagram to show a function. If not, justify your reasoning.

Yes. See diagram.
1. Isaac Messi is disorganized. To encourage Isaac to be more organized, his father promised to give him three dollars for every day that his room is clean and his schoolwork is organized.

**Part A:** Define the input and output in the given scenario.

**Input:**
Number of days Isaac cleans his room and schoolwork organized.

**Output:**
Amount of money Isaac earns

**Part B:** Write a function to model this situation.

Let \( d \) represent the number of days Isaac keeps his room clean and schoolwork organized.

\[ f(d) = 3d \]
2. The cost to manufacture \( x \) pairs of shoes can be represented by the function \( C(x) = 63x \). Complete the statement about the function.

\[
\begin{array}{c|c|c}
0 & \text{pairs of shoes cost} & 6. \\
6 & \text{} & 189. \\
63 & \text{} & 378. \\
378 & \text{} & 2,268. \\
\end{array}
\]

If \( C(6) = 378 \), then 6 pairs of shoes cost $378.

3. Which of the following relations is not a function?

A \( \{(0, 5), (2, 3), (5, 8), (3, 8)\} \)
B \( \{(4, 2), (−4, 5), (0, 0)\} \)
C \( \{(6, 5), (4, 1), (−3, 2), (4, 2)\} \)
D \( \{(-3, -3), (2, 1), (5, -2)\} \)

Answer: C
A ball is thrown into the air with an initial velocity of 15 meters per second. The quadratic function 
\[ h(t) = -4.9t^2 + 15t + 4 \]
represents the height of the ball above the ground, in meters, with respect to time \( t \), in seconds.

Determine \( h(2) \) and explain what it represents.

The height of the ball after 2 seconds: \( h(2) = 14.4 \) meters.

Would \(-3\) be a reasonable input for the function?

No, because input is time.

The graph below represents the height of the ball with respect to time.

![Graph of Height of the Ball Over Time](image)

What is a reasonable domain for the function?
\[ \{t | 0 \leq t \leq 3.5\} \text{ seconds} \]

What is a reasonable range for the function?
\[ \{h(t) | 0 \leq h(t) \leq 15.3\} \text{ meters} \]
Let's Practice!

1. On the moon, the time, in seconds, it takes for an object to fall a distance, \(d\), in feet, is given by the function \(f(d) = 1.11\sqrt{d}\).

   a. Determine \(f(5)\) and explain what it represents.

   \[h(2) = 1.11\sqrt{5} \approx 2.5 \text{ seconds}.
   \]

   b. The South Pole-Aitken basin on the moon is 42,768 feet deep. Determine a reasonable domain for a rock dropped from the rim of the basin.

   \[\{d|0 \leq d \leq 42768\} \text{ feet}\]

2. Floyd drinks two Mountain Dew sodas in the morning. The function that represents the amount of caffeine, in milligrams, remaining in his body after drinking the sodas is given by \(f(t) = 110(0.8855)^t\) where \(t\) is time in hours. Floyd says that in two days the caffeine is completely out of his system. Do you agree? Justify your answer.

   \(f(48) = 110(0.8855)^{48} \approx 0.32 \text{ milligrams}\)

   No, there is still 0.32 milligrams remaining.
3. Medical professionals say that 98.6°F is the normal body temperature of an average person. Healthy individuals’ temperatures should not vary more than 0.5°F from that temperature.

a. Write an absolute value function \( f(t) \) to describe an individual’s variance from normal body temperature, where \( t \) is the individual’s current temperature.

\[
f(t) = |t - 98.6|
\]

b. Determine \( f(101.5) \) and describe what that tells you about the individual.

\[
f(101.5) = 2.9
\]

The individual is sick because his/her temperature is 2.9°F from normal body temperature.

c. What is a reasonable domain for a healthy individual?

\[
\{t | 98.1 \leq t \leq 99.1\}^\circ F
\]
1. The length of a shipping box is two inches longer than the width and four times the height.

**Part A:** Write a function $V(w)$ that models the volume of the box, where $w$ is the width, in inches.

$$V(w) = (w + 2)(w)\left(\frac{w+2}{4}\right) = \frac{1}{4}(w)(w + 2)^2$$

**Part B:** Evaluate $V(10)$. Describe what this tells you about the box.

$$V(10) = 360 \text{ in}^3$$
A box that is 10 inches wide has a volume of 360 cubic inches

---

Want some help? You can always ask questions on the Algebra Wall and receive help from other students, teachers, and Study Experts. You can also help others on the Algebra Wall and earn Karma Points for doing so. Go to AlgebraNation.com to learn more and get started!
Let \( h(x) = 2x^2 + x - 5 \) and \( g(x) = -3x^2 + 4x + 1 \).

Find \( h(x) + g(x) \).

\[
(2x^2 + x - 5) + (-3x^2 + 4x + 1) \\
= 2x^2 - 3x^2 + x + 4x - 5 + 1 \\
= -x^2 + 5x - 4
\]

Find \( h(x) - g(x) \).

\[
(2x^2 + x - 5) - (-3x^2 + 4x + 1) \\
= 2x^2 + x - 5 + 3x^2 - 4x - 1 \\
= 2x^2 + 3x^2 + x - 4x - 5 - 1 \\
= 5x^2 - 3x - 6
\]
Let’s Practice!

1. Consider the following functions.

\[ f(x) = 3x^2 + x + 2 \]
\[ g(x) = 4x^2 + 2(3x - 4) \]
\[ h(x) = 5(x^2 - 1) \]

a. Find \( f(x) - g(x) \).

\[ (3x^2 + x + 2) - (4x^2 + 2(3x - 4)) \]
\[ = 3x^2 + x + 2 - 4x^2 - 6x + 8 \]
\[ = 3x^2 - 4x^2 + x - 6x + 2 + 8 \]
\[ = -x^2 - 5x + 10 \]

b. Find \( g(x) - h(x) \).

\[ 4x^2 + 2(3x - 4) - 5(x^2 - 1) \]
\[ = 4x^2 + 6x - 8 - 5x^2 + 5 \]
\[ = 4x^2 - 5x^2 + 6x - 8 + 5 \]
\[ = -x^2 + 6x - 3 \]
Try It!

2. Recall the functions we used earlier.

\[ f(x) = 3x^2 + x + 2 \]
\[ g(x) = 4x^2 + 2(3x - 4) \]
\[ h(x) = 5(x^2 - 1) \]

a. Let \( m(x) \) be \( f(x) + g(x) \). Find \( m(x) \).

\[ m(x) = (3x^2 + x + 2) + (4x^2 + 2(3x - 4)) \]
\[ m(x) = (3x^2 + x + 2) + (4x^2 + 6x - 8) \]
\[ m(x) = 3x^2 + x + 2 + 4x^2 + 6x - 8 \]
\[ m(x) = 3x^2 + 4x^2 + x + 6x + 2 - 8 \]
\[ m(x) = 7x^2 + 7x - 6 \]

b. Find \( h(x) - m(x) \).

\[ 5(x^2 - 1) - (7x^2 + 7x - 6) \]
\[ 5x^2 - 5 - 7x^2 - 7x + 6 \]
\[ 5x^2 - 7x^2 - 7x - 5 + 6 \]
\[ -2x^2 - 7x + 1 \]
1. Consider the functions below.

\[ f(x) = 2x^2 + 3x - 5 \]
\[ g(x) = 5x^2 + 4x - 1 \]

Which of the following is the resulting polynomial when \( f(x) \) is subtracted from \( g(x) \)?

A. \(-3x^2 - x - 4\)
B. \(-3x^2 + 7x - 6\)
C. \(3x^2 + x + 4\)
D. \(3x^2 + 7x - 6\)

**Answer:** C
Section 3 – Topic 4  
Multiplying Functions  

Use the distributive property and modeling to perform the following function operations.

Let \( f(x) = 3x^2 + 4x + 2 \) and \( g(x) = 2x + 3 \).

Find \( f(x) \cdot g(x) \).

\[
(3x^2 + 4x + 2)(2x + 3) \\
3x^2 \cdot 2x + 3x^2 \cdot 3 + 4x \cdot 2x + 4x \cdot 3 + 2 \cdot 2x + 2 \cdot 3 \\
6x^3 + 9x^2 + 8x^2 + 12x + 4x + 6 \\
6x^3 + 17x^2 + 16x + 6
\]

\[
\begin{array}{ccc}
3x^2 & 4x & 2 \\
2x & 6x^3 & 8x^2 & 4x \\
3 & 9x^2 & 12x & 6
\end{array}
\]

\[
6x^3 + 17x^2 + 16x + 6
\]
Let \( m(y) = 3y^5 - 2y^2 + 8 \) and \( p(y) = y^2 - 2 \).

Find \( m(y) \cdot p(y) \).

\[
\begin{array}{ccccccc}
3y^5 & 0y^4 & 0y^3 & -2y^2 & 0y & 8 \\
\hline
y^2 & 3y^7 & 0 & 0 & -2y^4 & 0 & 8y^2 \\
0y & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -6y^5 & 0 & 0 & 4y^2 & 0 & -16 \\
\end{array}
\]

\( 3y^7 - 6y^5 - 2y^4 + 12y^2 - 16 \)
Let's Practice!

1. Let \( h(x) = x - 1 \) and \( g(x) = x^3 + 6x^2 - 5 \).

Find \( h(x) \cdot g(x) \).

\[
(x^3 + 6x^2 - 5)(x - 1)
= x^3 \cdot x + x^3 \cdot (-1) + 6x^2 \cdot x + 6x^2 \cdot (-1) - 5 \cdot x - 5 \cdot (-1)
= x^4 - x^3 + 6x^3 - 6x^2 - 5x + 5
= x^4 + 5x^3 - 6x^2 - 5x + 5
\]

<table>
<thead>
<tr>
<th></th>
<th>( x^3 )</th>
<th>( 6x^2 )</th>
<th>( 0 \times )</th>
<th>( -5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( x^4 )</td>
<td>( 6x^3 )</td>
<td>( 0 )</td>
<td>( -5x )</td>
</tr>
<tr>
<td>( -1 )</td>
<td>( -x^3 )</td>
<td>( -6x^2 )</td>
<td>( 0 )</td>
<td>( 5 )</td>
</tr>
</tbody>
</table>

\[
x^4 + 5x^3 - 6x^2 - 5x + 5
\]
2. The envelope below has a mailing label:

\[ L(x) = 6x + 5 \]
\[ W(x) = 6x + 5 \]
\[ M(x) = x + 4 \]
\[ N(x) = x + 2 \]

a. Let \( A(x) = L(x) \cdot W(x) - M(x) \cdot N(x) \). Find \( A(x) \).

\[
L(x) \cdot W(x) = (6x + 5)(6x + 5) = 6x \cdot 6x + 6x \cdot 5 + 5 \cdot 6x + 5 \cdot 5 = 36x^2 + 30x + 30x + 25 = 36x^2 + 60x + 25
\]

\[
M(x) \cdot N(x) = (x + 4)(x + 2) = x \cdot x + x \cdot 2 + 4 \cdot x + 4 \cdot 2 = x^2 + 2x + 4x + 8 = x^2 + 6x + 8
\]

\[
A(x) = L(x) \cdot W(x) - M(x) \cdot N(x) = (36x^2 + 60x + 25) - (x^2 + 6x + 8) = 35x^2 + 54x + 17
\]
b. What does the function $A(x)$ represent in this problem?

The area of the front of the envelope excluding the address label.
1. A square has sides of length $s$. A rectangle is six inches shorter and eight inches wider than the square.

**Part A:** Express both the length and the width of the rectangle as a function of a side of the square.

**Length:** $L(s) = s - 6$

**Width:** $W(s) = s + 8$

**Part B:** Write a function to represent the area of the rectangle in terms of the sides of the square.

$$A(s) = (s - 6)(s + 8) = s^2 + 8s - 6s - 48 = s^2 + 2s - 48$$
Felicia needs to find the area of a rectangular field in her backyard. The length is represented by the function $L(x) = 4x^4 - 3x^2 + 6$ and the width is represented by the function $W(x) = x + 1$. Which of the following statements is correct about the area, $A(x)$, of the rectangular field in Felicia’s backyard? Select all that apply.

- $A(x) = 2[L(x) + W(x)]$
- The resulting expression for $A(x)$ is a fifth-degree polynomial.
- The resulting expression for $A(x)$ is a polynomial with a leading coefficient of 5.
- The resulting expression for $A(x)$ is a binomial with a constant of 6.
- $W(x) = \frac{A(x)}{L(x)}$

Want some help? You can always ask questions on the Algebra Wall and receive help from other students, teachers, and Study Experts. You can also help others on the Algebra Wall and earn Karma Points for doing so. Go to AlgebraNation.com to learn more and get started!
When we add two integers, what type of number is the sum?

**Integer**

When we multiply two irrational numbers, what type of number is the product?

**It could be rational or irrational.**

\[
\sqrt{2} \cdot \sqrt{8} = \sqrt{16} = 4 \\
\sqrt{2} \cdot \sqrt{3} = \sqrt{6}
\]

A set is **closed** for a specific operation if and only if the operation on two elements of the set *always* produces an element of the same set.

Are integers closed under addition? Justify your answer.

**Yes, the sum of integers always results in an integer.**

Are irrational numbers closed under multiplication? Justify your answer.

**No, the product of irrational numbers is not always irrational.**

Are integers closed under division? Justify your answer.

**No, the quotient of two integers does not always result in an integer.**
Let’s apply the closure property to polynomials.

Are the following statements true or false? If false, give a counterexample.

Polynomials are closed under addition.  
**True**

Polynomials are closed under subtraction.  
**True**

Polynomials are closed under multiplication.  
**True**

Polynomials are closed under division.

**False.** Consider the polynomials $2x^2$ and $x^5$.  
$$
\frac{2x^2}{x^5} = \frac{2}{x^3}.
$$

$\frac{2}{x^3}$ is not a polynomial.
Let’s Practice!

1. Check the boxes for the following sets that are closed under the given operations.

<table>
<thead>
<tr>
<th>Set</th>
<th>+</th>
<th>−</th>
<th>×</th>
<th>÷</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0, 1, 2, 3, 4, ... }</td>
<td>✗</td>
<td></td>
<td>✗</td>
<td></td>
</tr>
<tr>
<td>{..., −4, −3, −2, −1}</td>
<td>✗</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{..., −3, −2, −1, 0, 1, 2, 3, ... }</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
<td></td>
</tr>
<tr>
<td>{rational numbers}</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
<td></td>
</tr>
<tr>
<td>{polynomials}</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
<td></td>
</tr>
</tbody>
</table>
2. Ms. Sanabria claims that the closure properties for polynomials are analogous to the closure properties for integers. Mr. Roberts claims that the closure properties for polynomials are analogous to the closure properties of rational numbers. Who is correct? Explain your answer.

Ms. Sanabria and Mr. Roberts are both correct. Integers and polynomials are both closed under addition, subtraction, and multiplication. Neither are closed under division. AND polynomials and rational numbers are both closed under addition, subtraction, and multiplication. Neither are closed under division.

Rational numbers are not closed under division because zero is a rational number and division by zero is not rational.
BEAT THE TEST!

1. Choose from the following words and expressions to complete the statement below.

\[
2x^5 + (3y)^{-2} - 2 \quad (5y)^2 + 4x + 3y^3 \\
5y^{-1} + 7x^2 + 8y^2
\]

integers \quad variables \quad whole numbers
coefficients \quad rational \quad numbers
whole numbers

The product of \(5x^4 - 3x^2 + 2\) and \((5y)^2 + 4x + 3y^3\) illustrates the closure property because the _exponents_ of the product are _whole numbers_ and the product is a polynomial.
Section 3 – Topic 6
Real-World Combinations and Compositions of Functions

There are many times in real world situations when we must combine functions. Profit and revenue functions are a great example of this.

Let’s Practice!

1. At the fall festival, the senior class sponsors hayrides to raise money for the senior trip. The ticket price is $5.00 and each hayride carries an average of 15 people. They consider raising the ticket price in order to earn more money. For each $0.50 increase in price, an average of 2 fewer seats will be sold. Let $x$ represent the number of $0.50 increases.

   a. Write a function, $T(x)$, to represent the cost of one ticket based on the number of increases.
      
      $$T(x) = 5 + 0.50x$$

   b. Write a function, $R(x)$, to represent the number of riders based on the number of increases.
      
      $$R(x) = 15 - 2x$$

   c. Write a revenue function for the hayride that could be used to maximize revenue.
      
      $$T(x) \cdot R(x) = (5 + 0.50x)(15 - 2x)$$
Try It!

2. The freshman class is selling t-shirts to raise money for a field trip. The cost of each custom designed t-shirt is $8. There is a $45 fee to create the design. The class plans to sell the shirts for $12.

a. Define the variable.

Let \( x \) represent the number of shirts sold.

b. Write a cost function.

\[
C(x) = 45 + 8x
\]

c. Write a revenue function.

\[
R(x) = 12x
\]

d. Write a profit function.

\[
P(x) = R(x) - C(x) = 12x - (45 + 8x) \\
= 4x - 45
\]
Let’s Practice!

3. Priscilla works at a cosmetics store. She receives a weekly salary of $350 and is paid a 3% commission on weekly sales over $1500.

a. Let $x$ represent Priscilla’s weekly sales. Write a function, $f(x)$, to represent Priscilla’s weekly sales over $1500$.

$$f(x) = x - 1500$$

b. Let $x$ represent the weekly sales on which Priscilla earns commission. Write a function, $g(x)$, to represent Priscilla’s commission.

$$g(x) = 0.03x$$

c. Write a composite function, $(g \circ f)(x)$ to represent the amount of money Priscilla earns on commission.

$$(g \circ f)(x) = 0.03(x - 1500) = 0.03x - 45$$
4. A landscaping company installed a sprinkler that rotates and sprays water in a circular pattern. The water reaches its maximum radius of 10 feet after 30 seconds. The company wants to know the area that the sprinkler is covering at any given time after the sprinkler is turned on.

a. Let $t$ represent the time in seconds after the sprinkler is turned on. Write a function, $r(t)$, to represent the size of the growing radius based on time after the sprinkler is turned on.

   Since the radius grows to 10 feet in 30 seconds, it is growing at a rate of $\frac{1}{3}$ foot per second.
   
   $$r(t) = \frac{1}{3}t$$

b. Let $r$ represent the size of the radius at any given time. Write a function, $A(r)$, to represent the area that the sprinkler covers at any given time, in seconds.

   $$A(r) = \pi r^2$$

c. Write a composite function, $A(r(t))$ to represent the area based on the time, in seconds, after the sprinkler is turned on.

   $$A(r(t)) = \pi \left( \frac{1}{3}t \right)^2 = \frac{1}{9} t^2 \pi$$
1. A furniture store charges 6.5% sales tax on the cost of the furniture and a $20 delivery fee. (The delivery fee is not subject to sales tax.)

The following functions represent the situation:
\[ f(a) = 1.065a \]
\[ g(b) = b + 20 \]

**Part A:** Write the function \( g(f(a)) \).

\[ g(f(a)) = 1.065a + 20 \]

**Part B:** Match each of the following to what they represent. Some letters will be used twice.

- \( a \) C A. The cost of the furniture, sales tax, and delivery fee.
- \( b \) B B. The cost of the furniture and sales tax.
- \( f(a) \) B C. The cost of the furniture.
- \( g(b) \) A
- \( g(f(a)) \) A
Section 3 – Topic 7
Key Features of Graphs of Functions – Part 1

Let’s review the definition of a function.

Every input value \((x)\) corresponds to _____ only one _____ output value \((y)\).

Consider the following graph.

How can a vertical line help us quickly determine if a graph represents a function?

Any vertical line would only cross the graph ONCE. If you can draw any vertical line that crosses the graph two or more times, it is NOT function.
We call this the **vertical line test**. Use the vertical line test to determine if the graph above represents a function.

**Important facts:**

- Graphs of lines are not always functions. Can you describe a graph of a line that is not a function?

  **A vertical line would not be a function.**

- Functions are not always linear.

Sketch a graph of a function that is not linear.

  **Sample Answer**
Let's Practice!

1. Use the vertical line test to determine if the following graphs are functions.
Try It!

2. Which of the following graphs represent functions? Select all that apply.

```
Not a function
Is a function
```

Consider the following scenarios. Determine if each one represents a function or not.

a. An analyst takes a survey of people about their heights (in inches) and their ages. She then relates their heights to their ages (in years).

   Not a function

b. A geometry student is dilating a circle and analyzes the area of the circle as it relates to the radius.

   Is a function
c. A teacher has a roster of 32 students and relates the students’ letter grades to the percent of points earned.

**Is a function**

d. A boy throws a tennis ball in the air and lets it fall to the ground. The boy relates the time passed to the height of the ball.

**Is a function**

It’s important to understand key features of graphs.

- An **x-intercept** of a graph is the location where the graph crosses the **x-axis**.

- The **y-coordinate** of the **x-intercept** is always **zero**.

- The **y-intercept** of a graph is the location where the graph crosses the **y-axis**.

- The **x-coordinate** of the **y-intercept** is always **zero**.

- The **x-intercept** is the **solution** to a function or graph.
All of these features are very helpful in understanding real-world context.

Let’s Practice!

3. Consider the following graph that represents the height, in feet, of a water balloon dropped from a 2nd story window after a given number of seconds.
a. What is the $x$-intercept?
(1.25, 0)

b. What is the $y$-intercept?
(0, 25)

c. Label the intercepts on the graph.

Try It!

4. Refer to the previous problem for the following questions.
a. What does the $y$-intercept represent in this real-world context?

The height from which the water balloon was dropped.

b. What does the $x$-intercept represent in this real-world context?

The number of seconds it took the water balloon to hit the ground after being dropped.

c. What is the solution to this situation?

1.25 seconds

Section 3 – Topic 8

Key Features of Graphs of Functions – Part 2

Let’s discuss other key features of graphs of functions.
- **Domain**: the input or the ____ values.

- **Range**: the ________________ or the \( y \)-values.

- **Increasing intervals**: as the \( x \)-values ____________, the \( y \)-values ____________.

- **Decreasing intervals**: as the \( x \)-values ____________, the \( y \)-values ____________.

- **Relative maximum**: the point on a graph where the interval changes from ________ to ________.

- **Relative minimum**: the point on a graph where the interval changes from ________ to ________.

Let’s Practice!

1. Use the following graph of an **absolute value function** to answer the questions below.

**STUDY EDGE TIP**

We read a graph from left to right to determine if it is increasing or decreasing, like reading a book.
a. Define the domain.

**All real numbers.** \( \{x | x \in \mathbb{R}\} \)

b. Define the range.

\( \{y | y \geq 0\} \)

c. Where is the graph increasing?

\( \{x | x > 0\} \)

d. Where is the graph decreasing?

\( \{x | x < 0\} \)

e. Identify any relative maximums.

**There are none.**

f. Identify any relative minimums.

(0, 0)

Try It!

2. Use the graph of the following **quadratic function** to answer the questions below.
a. Define the domain.
   **All real numbers.** \( \{x | x \in \mathbb{R}\} \)

b. Define the range.
   \( \{y | y \leq 0\} \)

c. Where is the graph increasing?
   \( \{x | x < 0\} \)

d. Where is the graph decreasing?
   \( \{x | x > 0\} \)

e. Identify any relative maximums.
   \((0, 0)\)

f. Identify any relative minimums.
   **There are none.**

3. Describe everything you know about the key features of the following graph of an **exponential function**.
The domain is \( \{x | x \in \mathbb{R}\} \).
The range is \( \{y | y > 0\} \).
The graph is increasing for all real numbers of \( x \).
The graph is never decreasing.
The graph has no relative maximums or minimums.

1. The following graph is a piecewise function.
Which of the following statements are true about the graph? Select all that apply.

- The graph is increasing when the domain is 
  \(-6 < x < -4\).
- The graph has exactly one relative minimum.
- The graph is increasing when 
  \(-4 \leq x \leq 0\).
- The graph is increasing when \(x > 4\).
- The graph is decreasing when the domain is 
  \(\{x | x < -6 \cup x > 2\}\).
- The range is \(\{y | 0 \leq y < 4 \cup y \geq 5\}\).
- There is a relative minimum at \((2, 2)\).

---

Want some help? You can always ask questions on the Algebra Wall and receive help from other students, teachers, and Study Experts. You can also help others on the Algebra Wall and earn Karma Points for doing so. Go to AlgebraNation.com to learn more and get started!

**Section 3 – Topic 9**

**Average Rate of Change Over an Interval**
Consider the following graph of the square root function \( f(x) = \sqrt{x} \).

Draw a line connecting \( a \) and \( b \).

Determine the slope of the line between the interval \([a, b]\).

\[(0, 0) \text{ to } (4, 2)\]

\[
\frac{2-0}{4-0} = \frac{2}{4} = \frac{1}{2}
\]

For every two points \( x_1 \) and \( x_2 \), where \( x_1 \neq x_2 \), \((x_1, y_1)\) and \((x_2, y_2)\) form a straight line and create a ______________ function.

To determine the average rate of change for any function \( f(x) \) over an interval, we can use two points \((x_1, f(x_1))\) and \((x_2, f(x_2))\) that lie on that interval.

The process to find the slope of a linear function is:

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

We can also use the slope formula to find the average rate of change over an interval \([a, b]\), where \( x_1 = a \) and \( x_2 = b \).

\[
\frac{f(b) - f(a)}{b - a}
\]
Let’s Practice!

1. Tom is jumping off the diving board at the Tony Dapolito Pool. His height is modeled by the quadratic function \( h(t) = -t^2 + 2t + 4 \), where \( h(t) \) represents height above water (in feet), and \( t \) represents time after jumping (in seconds).

![Tom's Jump into Pool](image)

**a.** Determine the average rate of change for the following intervals.

\[
\begin{align*}
[a, b] & \quad m = 1 \\
[b, c] & \quad m = -1 \\
[c, d] & \quad m = -\frac{1}{0.31}
\end{align*}
\]

**b.** Compare Tom's average rate of change over the interval \([a, b]\) with his average rate of change over the interval \([b, c]\). What does this represent in real life?

\([a, b]\) when he was going up
\([b, c]\) when he was going down
2. Consider the table for the exponential function, $p(x) = 3^x$, shown below.

<table>
<thead>
<tr>
<th>Point</th>
<th>$x$</th>
<th>$p(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$N$</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$R$</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>$T$</td>
<td>3</td>
<td>27</td>
</tr>
</tbody>
</table>

a. Determine the average rate of change over the interval $[N, T]$.

$$\frac{27-3}{3-1} = \frac{24}{2} = 12$$

b. Compare the average rate of change over the interval $[M, N]$ with the average rate of change over the interval $[R, T]$.

For $[M, N]$ we have $\frac{3-1}{1-0} = \frac{2}{1} = 2$

For $[R, T]$ we have $\frac{27-9}{3-2} = \frac{18}{1} = 18$
3. Determine the intervals that have the same average rate of change in the graph \( j(x) = \sqrt[3]{x} \) below.

\[
[a, b] = [d, e] = \frac{1}{7}; [a, c] = [a, e] = [c, e] = \frac{1}{4}; [a, d] = [b, e] = \frac{1}{3}; [b, c] = [b, d] = [c, d] = 1
\]
1. Suppose that the cost of producing $r$ radios is defined by $c(r) = 300 + 15r - 0.3r^2$. Determine which of the following intervals has the greatest average rate of change for the cost to produce a radio.

A. Between 20 and 25 radios.
B. Between 60 and 65 radios.
C. Between 5 and 10 radios.
D. Between 30 and 35 radios.

Answer is B.
2. Consider the absolute value function $f(x)$ and the step function $g(x)$ in the graphs below.

Which of the following is true about the rate of change of the graphs?

A. The average rate of change for $f(x)$ over the interval $[b, c]$ is greater than the average rate of change for $g(x)$ over the interval $[j, k]$.

B. The average rate of change for $f(x)$ over the interval $[a, c]$ is greater than the average rate of change for $g(x)$ over the interval $[d, r]$.

C. The average rate of change for $f(x)$ over the interval $[a, b]$ is $-\frac{1}{2}$.

D. The average rate of change for $g(x)$ over the interval $[d, j]$ is $-\frac{1}{2}$.

Answer is A.
Section 3 – Topic 10
Transformations of Functions

The graph of $f(x)$ is shown below.
The following graphs are transformations of $f(x)$. Describe what happened in each graph.

$f(x) + 2$  

The graph shifted up 2

$f(x) - 1$  

The graph shifted down 1

$f(x + 2)$  

The graph shifted left 2

$f(x - 1)$  

The graph shifted right 1

Which graphs transformed the independent variable?  
$f(x + 2)$ and $f(x - 1)$

Which graphs transformed the dependent variable?  
$f(x) + 2$ and $f(x) - 1$
Let’s Practice!

1. For the following functions, state whether the independent or dependent variable is being transformed and describe the transformation (assume $k > 0$).

   a. $f(x) + k$
      
      **Dependent Variable** $(y)$
      The graph shifts up $k$ units.

   b. $f(x) - k$
      
      **Dependent Variable** $(y)$
      The graph shifts down $k$ units.

   c. $f(x + k)$
      
      **Independent Variable** $(x)$
      The graph shifts left $k$ units.

   d. $f(x - k)$
      
      **Independent Variable** $(x)$
      The graph shifts right $k$ units.
The following table represents the function \( g(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>0.25</td>
</tr>
<tr>
<td>(-1)</td>
<td>0.5</td>
</tr>
<tr>
<td>(0)</td>
<td>1</td>
</tr>
<tr>
<td>(1)</td>
<td>2</td>
</tr>
<tr>
<td>(2)</td>
<td>4</td>
</tr>
</tbody>
</table>

The function \( h(x) = g(2x) \). Complete the table for \( h(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(2x) )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>( g(2(-1)) )</td>
<td>0.25</td>
</tr>
<tr>
<td>(-0.5)</td>
<td>( g(2(-0.5)) )</td>
<td>0.5</td>
</tr>
<tr>
<td>(0)</td>
<td>( g(2 \cdot 0) )</td>
<td>1</td>
</tr>
<tr>
<td>(0.5)</td>
<td>( g(2 \cdot 0.5) )</td>
<td>2</td>
</tr>
<tr>
<td>(1)</td>
<td>( g(2 \cdot 1) )</td>
<td>4</td>
</tr>
</tbody>
</table>
3. The table below shows the values for the function $f(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$4$</td>
<td>$2$</td>
<td>$0$</td>
<td>$2$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

Complete the table for the function $-\frac{1}{2}f(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\frac{1}{2}f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$2$</td>
<td>$-2$</td>
</tr>
</tbody>
</table>
Let \( g(x) = f(x + 3) - 2 \).

Graph \( g(x) \) on the coordinate plane with \( f(x) \).
1. The graph of $f(x)$ is shown below.

Let $g(x) = f(x - 3)$ and $h(x) = f(x) - 3$.

Graph $g(x)$ and $h(x)$ on the coordinate plane with $f(x)$. 

![Graph of $f(x)$, $g(x)$, and $h(x)$]
2. The table below shows the values for the function \( p(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-4)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

Complete the table for the function \( \frac{1}{2}p(x) - 3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{1}{2}p(x) - 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-4)</td>
<td>3</td>
</tr>
<tr>
<td>(-1)</td>
<td>0</td>
</tr>
<tr>
<td>(0)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(2)</td>
<td>1</td>
</tr>
<tr>
<td>(3)</td>
<td>2</td>
</tr>
</tbody>
</table>

Great job! You have reached the end of this section. Now it’s time to try the “Test Yourself! Practice Tool,” where you can practice all the skills and concepts you learned in this section. Log in to Algebra Nation and try out the “Test Yourself! Practice Tool” so you can see how well you know these topics!